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FULL NASH IMPLEMENTATION OF NEUTRAL SOCIAL FUNCTIONS

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Abstract

This paper characterizes neutral social functions that are fully implementable. A necessary condition for full implementation under either the Nash equilibrium concept or the strong Nash equilibrium concept is that the neutral social function being implemented be monotonic and simple. If a neutral monotonic social function is simple and the set of winning coalitions is nondictatorial then the social function is fully implementable by a set of Nash equilibria. For finite alternative sets a neutral monotonic social function will be fully implementable by a set of strong Nash equilibria if and only if it is simple and dictatorial.

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I. INTRODUCTION

Two problems have been prominent in social choice theory in recent years. One is the problem of determining the restrictions that will result if social choice processes are required to have certain qualities when individual preferences are known. The qualities that have been studied are possible ethical or logical desiderata such as neutrality, monotonicity, transitivity and acyclicity. The restrictions usually involve the delineation of which coalitions or individuals in a society can be permitted to determine social choices.

A second problem, the problem of "implementation," arises from the observation that individual preferences generally are not known and that individuals may have an incentive not to reveal their true preferences given the social choice consequences. Unless this second problem is solved, solution to the first problem of "characterizing social choice processes" may not be of any practical import. It does little good to know that a particular social choice process has certain ethical or logical qualities if everyone reveals their true preferences when, in fact, there is a significant incentive for individuals to try to change the outcome by lying.

Recent work such as Roberts (1979) and Sen (1983) has attempted to treat the implementation problem in the same manner as the treatment of the first problem. I.e., this work has attempted to characterize social choice processes for which the implementation problem can be solved. This article continues that characterization task.

As can be seen from the recent surveys in Dasgupta, Hammond and Maskin (1979) and Moulin (1981), results on the implementation problem generally have been negative, describing broad classes of social choice rules that cannot be implemented. A striking exception is Maskin (1977). In that paper, Maskin showed that for finite alternative sets social choice correspondences satisfying two conditions, "Maskin monotonicity" and "NVP," can be fully implemented by Nash equilibria. (The Maskin monotonicity condition is a necessary condition while NVP is not.) In Maskin (1979), it is demonstrated that this result is sensitive to the equilibrium concept employed. In particular, for finite alternative sets, social choice correspondences that are fully implementable by strong Nash equilibria must be Maskin monotonic but must <u>not</u> satisfy NVP.

Maskin monotonicity requires that an alternative x in the choice set of a social choice correspondence will remain in the choice set if individual preferences change in such a way that no person who found x at least as good as some other alternative y prior to the change finds y better than x after the change. This condition, which is a necessary condition for full implementation by both Nash and strong Nash equilibria, speaks to the composition of the choice set when preferences change. It has no obvious connection to restrictions on which coalitions can dictate social choice, the traditional restrictions examined in characterizing social choice processes. One of the main tasks in this article is to demonstrate such a connection.

The condition NVP requires that if all but one person in society finds an alternative y to be at least as good as any other, y must be in the choice set for the social choice correspondence. This condition is more "coalitional" in content than Maskin monotonicity. Using the results on Maskin monotonicity, this article translates NVP directly into familiar coalitional conditions.

After definitions and notation are developed in Section II, Section III extends the Maskin (1977) result that under Maskin monotonicity and NVP social choice correspondences are fully implementable by Nash equilibria to alternative sets that are "CCE subsets," compact, convex subsets of a Euclidean space. This extension allows many of the characterization results to apply whether the alternative set is a CCE subset or finite. In addition, it allows comparison of the CCE subset results here with other CCE subset characterization results in the literature.

The main results are in Section IV. This section begins by limiting consideration to social choice correspondences that are neutral social functions, i.e. neutral with the "socially preferred to" relation being an asymmetric binary relation.

A key result in Section IV is Theorem 5: a neutral social

function is Maskin monotonic if and only if it is a simple neutral monotonic social function. Thus, Maskin monotonicity, apparently a condition concerning choice set elements, produces two "coalitional" conditions, monotonicity and simplicity. Monotonicity requires that when members are added to a winning coalition, it still wins, and when members are subtracted from a losing coalition, it still loses.

Simplicity requires that there is a set S of coalitions such that for any two alternatives x and y, x is socially preferred to y if and only if all the members in a coalition in S so prefer. This condition is quite restrictive since it rules out all social choice rules where the set of coalitions that win depends on who is indifferent. For example, Pareto extension rules (x is socially preferred to y if at least one person so prefers and no one has the opposite preference) and relative majority rule (x is socially preferred to y if a majority of those who are not indifferent so prefer) are not simple.

For a simple neutral monotonic social function, NVP is equivalent to the set of winning coalitions not being an ultrafilter. I.e., NVP requires that the social function is not dictatorial.

Combining these results with those of Maskin (1977) and Maskin (1979) yields a cluster of characterization theorems. Neutral social functions will be fully Nash implementable or fully strong Nash implementable only if they are simple neutral monotonic social functions. A neutral social function will be fully Nash implementable if it is a non-dictatorial simple neutral monotonic social function.

Each of these results holds both for finite alternative sets and CCE subset alternative sets.

The final major characterization result applies when the alternative set is finite. In that case neutral social functions will be fully strong Nash implementable if and only if they are dictatorial simple neutral monotonic social functions. This is in contrast to the result that full Nash implementation of neutral social functions is possible without a dictator. Imposing a stronger equilibrium concept has a significant cost.

The characterization results in this paper can be contrasted to other results in the literature across three basic parameters: restrictions on the class of social choice processes studied, restrictions on the equilibrium concepts studied, and restrictions on the implementation concepts studied. With respect to the later two parameters, this article is limited to Nash and strong Nash equilibrium concepts and to full implementation. The reason is that these are the restrictions on Maskin's results, and the positive nature of some of those results motivates the effort to characterize them in a "coalitional" way.

Ferejohn, Grether and McKelvey (1982) develop some characterization results when implementation rather than full implementation is the implementation concept. A full implementation analogy to one of their implementation results is developed as Corollary 2 in Section IV. Despite the linguistic implication that full implementation is somehow normatively better ("fuller") than

implementation, no such superiority actually exists. Both implementation concepts require that all equilibria be in the choice set. Full implementation requires that all choice set elements be reachable by some equilibrium while implementation requires only that at least one choice set alternative be reachable by an equilibrium. The only normative criteria arising from the social choice rule is that choice set elements are desirable. The fact that more choice set elements are reachable by one decision mechanism has no normative significance under the social choice rule since it does not distinguish between choice set elements.

Finally, this article restricts the domain of social choice rules to neutral social functions. Neutrality is a nontrivial ethical and structural assumption. It bars the use of separate processes for particular choices. Thus, for example, neutrality is violated if a higher voting requirement is required for certain "special" decisions (such as constitutional amendments) than for "ordinary" decisions.

The main justification for assuming neutralty lies in the fact that that assumption allows Maskin's positive results, highly unusual against the general backdrop of negative implementation results, to be characterized "coalitionally." But beyond that, much of the literature characterizing implementation results imposes restrictions that are arguably even more severe and that lead to highly negative results. For example, Roberts (1979) and Sen (1983) are two leading recent papers. Almost all of Roberts (1979) and all of Sen (1983) are limited to a study of decisive social choice correspondences, i.e., social choice correspondences that are single-valued. Roberts notes that decisive social choice correpondences are attractive because the social planner must pick one alternative and such social choice correspondences specify which one is most desirable. Decisiveness also eliminates the need to choose between full implementation and implementation. With only one member in the choice set those implementation concepts are equivalent.

Although decisiveness is a desirable trait for a social process, imposing decisiveness severely restricts the class of social choice rules under consideration and that restriction may involve the sacrifice of other ethical or logical desiderata. Consider, for example, proportional voting rules. Craven (1971) shows that for the choice set to be nonempty over a finite alternative set with any more than a few alternatives in it, a voting proportion close to unanimity is required. Greenberg (1979) makes a similar showing for CCE subset alternative sets with more than a few dimensions. But proportional voting rules requiring proportions significantly greater than one-half often will not be decisive. Two alternatives preferred to all others by everyone may "tie" as social choices because voters are nearly equally divided between them. For proportional voting rules, decisiveness therefore may be obtained only at the cost of the possibility of having an empty choice set.

Another problem with imposing decisiveness on the underlying social choice correspondences is that extremely negative results ensue. Theorem 3.4 in Sen (1983) as well as results in Maskin (1979)

and Dasgupta, Hammond and Maskin (1979) indicate that decisive social choice correspondences are implementable by strong Nash equilibria or Nash equilibria only if they are dictatorial. That result holds even if individual indifference is excluded as a possibility. This leads Sen (1983) to relax the implementation concept from implementable to what he calls "partially implementable." A single-valued social choice correspondence is partially implementable by a decision mechanism if that mechanism has at least one equilibrium that is the single member of the choice set. The cost of imposing this concept instead of more traditional implementation standards is carefully noted by Sen himself: there is a possibility that "inoptimal" alternatives will be selected as equilibria by the decision mechanism. One can only hope, along with Sen, that "with suitable 'rules of behavior' agreed upon by the individuals, their cooperative equilibrium may indeed lead to the selection of the (desired) outcome" (Sen (1983, p. 3)). Even if this problem with partial implementation is ignored, Theorem 4.11 and Corollary 4.16 in Sen (1983), the main results of that paper, indicate that the coalitional structure needs to contain a great deal of veto power in order to guarantee that a social choice correspondence is partially implementable by strong Nash equilibria.

Ultimately, the choice of restrictions is a value judgment. One may be willing to tolerate both the potential problems with partial implementation and the restrictions inherent in requiring decisiveness in order to use a strong Nash equilibrium concept, in order to have the quality of decisiveness and to avoid imposing neutrality. Nonetheless, the positive characterization results proven here for full Nash implementation are quite promising. The scope of the positive results compare favorably with the scope of other positive results in the literature, and the need to impose neutrality to attain the results here seems no more onerous than the compromises required to obtain other positive results.

II. DEFINITIONS, ASSUMPTIONS AND NOTATION

Take the set of social alternatives to be A. When A is finite, |A| is the number of members in A. When A is a subset of Euclidean space, d(A) denotes the dimension of A.

A <u>CCE subset</u> is a nonempty, compact, convex subset of a Euclidean space. When A is a CCE subset with d(A) = m, then A is an <u>m-CCE subset</u>. It is assumed that A contains more than one alternative, so that there is a nontrivial social choice problem. Whenever the nature of A is unspecified, it can be either finite or a CCE-subset.

P is a <u>preference relation</u> on A if P is an asymmetric binary relation on A, i.e., xPy and yPx cannot both be true for x, y ε A. If they are both false then xIy, and xRy means that xPy or xIy. When an individual i has tastes characterized by a preference relation, that preference relation shall be denoted by P_i, R_i, and I_i.

A preference relation is a <u>weak ordering</u> if R is transitive and complete. A preference relation is <u>acyclic</u> if for all $x_1, x_2, \ldots, x_n$ ε A $x_1Px_2, x_2Px_3, \ldots, x_{n-1}Px_n \Rightarrow x_1Rx_n$. The set U(A,P) =

$\{x \in A | xRy \ \forall \ y \in A\}$ is the set of <u>undominated alternatives</u> for A under P. A preference relation, P, is <u>continuous-valued</u> over a set of alternatives that is a subset, S, of Euclidean space if and only if for every x ε S:

(A) the set $\{y \in S | xPy\}$ is an open set;

and (B) the set $\{y \in S | yPx\}$ is an open set.

Suppose that A is a CCE subset of a Euclidean space, S. Then an individual, i, has <u>type I preferences</u> if there exists a "bliss point" x (in S but not necessarily in A) such that yP_iz if and only if alternative y is closer in Euclidean distance to x than alternative z.[1] When an individual can have type I preferences with any bliss point in S, then type I preferences are said to be <u>admissible</u> for that individual.

The set of all individuals in society is I. Throughout this article the set I will be taken to be finite with n members. W^I is the product space of individual weak orderings for the individuals in society. When A is a CCE-subset, it is assumed that type I preferences are admissible for all individuals.

A <u>social function</u> is a mapping from W^I into the set of all preference relations on A. A <u>social decision function</u> is a mapping from W^I into the set of all acyclic preference relations on A.

A <u>profile</u> is a member of W^I. For a profile p and a, b ε A, define p(a > b) as the set of individuals who prefer a to b. The <u>concerned set</u> for p and the pair of alternatives {a,b} is p(a > b) $\cup$ p(b > a). Let p and q be profiles and let a, b, c, d ε A.

A social function, σ, is *neutral and monotonic* when the following condition holds: if $p(a > b) \subseteq q(c > d)$ and $q(d > c) \subseteq p(b > a)$, then $a\sigma(p)b$ implies $c\sigma(q)d$. If the condition holds with equalities rather than inclusions, then σ is *neutral*. A *binary decision rule*, σ, is a social function that satisfies the following property of *binaryness*: if $p(a > b) = q(a > b)$ and $p(b > a) = q(b > a)$ then $a\sigma(p)b$ implies $a\sigma(q)b$. A social function, σ, is *anonymous* if and only if for any permutation γ of $(1,2,\ldots,n)$, $\sigma(p_1,p_2,\ldots,p_n) = \sigma(p_{\gamma(1)}, p_{\gamma(2)},\ldots,p_{\gamma(n)})$ where p_i is the weak ordering of individual i.

If J is a nonempty subset of I, then a *simple game* on J is a collection of subsets of J, Γ_J, such that:

(a) $A \in \Gamma_J$, $A \subseteq B \Rightarrow B \in \Gamma_J$;

(b) $A \in \Gamma_J \Rightarrow A^c \notin \Gamma_J$, where A^c is the complement of A in J.

When property (a) is true, but property (b) may or may not be true, Γ_J is called a *monotonic game*. When property (b) is true, but property (a) may or not be true, Γ_J is called a *proper game*. The *null (simple) game* on J is the empty collection of subsets of J. If Γ_J is a proper game on J, then $\Gamma_J^* = \{E \subseteq J \mid E^c \in \Gamma_J\}$ where E^c is the complement of E in J. Γ_J often is called the family of *winning coalitions* and Γ_J^* the family of *losing coalitions*. Note that under these definitions in a given proper game Γ_J, it may be true that a coalition $E \subseteq J$ is neither winning nor losing.

A *direct sum of simple games* is an indexed family $[\Gamma_J]_{J \in 2^I}$ such that:

(i) Γ_J is a simple game for J (possibly null);

(ii) for all K, L $\in 2^I$, if $K \subseteq L$, then $\Gamma_L \cap 2^K \subseteq \Gamma_K$;

(iii) for all K, L $\in 2^I$, if $K \subseteq L$, then $\Gamma_K^* \subseteq \Gamma_L^*$.

Every *collection of proper games*, $\Gamma = [\Gamma_J]_{J \in 2^I}$ where each Γ_J is a proper game, *generates an aggregation rule*, μ_Γ, where for every profile p and alternatives a, b $\in$ A, $a\mu_\Gamma(p)b$ if and only if $p(a > b) \in \Gamma_{p(a>b) \cup p(b>a)}$. I.e., the set of individuals who prefer a over b is a winning coalition in the proper game defined on the set of individuals concerned about a versus b. If σ is a social function and $J \subseteq K$, then J is said to be a *decisive subset for K* (with respect to σ) if for all profiles p where K is the concerned set, $J \subseteq p(a > b) \Rightarrow a\sigma(p)b$, for all a, b $\in$ A. The set of all decisive subsets for K is the *decisive set for K*. When a social function is generated by a direct sum of simple games, Γ, then Γ_J is the decisive set for $J \subseteq I$. A social function is *simple* if for any x, y $\in$ A, xPy if and only if all the members of at least one of the coalitions in the decisive set for I prefer x to y. When a simple social function is generated by Γ, a direct sum of simple games then for $J \subseteq I$ $\Gamma_J = \Gamma_I \cap 2^J$. A *prefilter* on J is a simple game, Γ_J, on J such that $\cap \Gamma_J \neq 0$ and Γ_J is not a null game. For a prefilter, Γ_J, on J, the set $\cap \Gamma_J$ is called the *collegium* for Γ_J. A *filter* on J is a prefilter, Γ_J, on J such that E, $F \in \Gamma_J \Rightarrow E \cap F \in \Gamma_J$. An *ultrafilter* on J is a filter, Γ_J, on J such that for all $E \subseteq J$ either $E \in \Gamma_J$ or $E^c \in \Gamma_J$ where E^c is the complement of E in J.

A *social choice correspondence* f is a mapping f: $W^I \rightarrow A$. A social choice correspondence may be multiple-valued. The image set of a social choice correspondence f under a profile p of weak orderings is denoted f(p) and is called the <u>choice set</u> of f for profile p. Let T be the set of all $p \in W^I$ such that there is some $j \in I$ and some $x_p \in A$ such that for all $i \in I/\{j\}$ $x_p P_i y$ for all $y \in A/\{x_p\}$ and there is at most one $z \in A/\{x_p\}$ such that $zR_j x_p$. A social choice correspondence is <u>minimally democratic</u> if for all $p \in T$ $f(p) = \{x_p\}$. I.e., for $|I| = n$ if x is at the top of n - 1 individuals' weak orderings and is at worst second in the remaining individual's weak ordering, then the choice set must consist of x alone.

Consider the mapping f': $W^I \rightarrow A$ with $f'(q) = U(A,P)$ where σ is a social function, $\sigma(q) = P$, a preference relation, and U(A,P) is the set of undominated alternatives in A under P. This mapping f' is a social choice correspondence. When this article speaks of a social function σ as a social choice correspondence, the mapping f' is meant.

Let $R_p(a,b) = \{i \in I | aR_i b\}$ under the profile p of weak orderings for the set I of all members of society and for a, $b \in A$. A social choice correspondence f is <u>Maskin monotonic</u> if for any profiles p and q of weak orderings and any $a \in A$, $a \in f(p) \Rightarrow a \in f(q)$ if $R_q(a,b) \supseteq R_p(a,b)$ for all $b \in A$. I.e., a social choice correspondence f is Maskin monotonic when for any transformation T: $p \rightarrow q$ of preference profiles, if a is in the choice set of f for p, then it is in the choice set of f for q if a is not demoted in any individual's weak ordering. <u>Not demoted</u> means that $aR_i b$ under p implies $aR_i b$ under q.

A social choice correspondence, f, has the property <u>NVP</u> if and only if for any profile p of individual weak orderings if there exists $i \in I$ and $a \in A$ such that for all $j \in I$ where $j \neq i$ $aR_j b$ for all $b \in A$ then $a \in f(p)$. I.e., if an alternative is at the top of n - 1 individuals' preference orderings, then the last individual cannot prevent the alternative from being in the choice set.

Let S_i denote a strategy space for individual i and let $S = \prod_{j=1}^{n} S_j$ be the product space of the individual strategy spaces. A <u>decision mechanism</u> d is a single-valued mapping from S to A. A decision mechanism is called a <u>direct mechanism</u> if $S_i = P^*$ or a subset of P^* for all $i \in I$, where P^* is the set of all possible preference relations over A. A decision mechanism is an <u>indirect mechanism</u> if for some $j \in I$ S_j is not P^* or any subset of P^*.

Let $\bar{s}$ denote $(\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n)$, i.e., all individuals choose their barred strategies. Let $\bar{s}/s_i$ denote $(\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_{i-1}, s_i, \bar{s}_{i+1}, \ldots, \bar{s}_n)$, i.e., all individuals choose barred strategies except individual i who chooses $s_i \in S_i$. Now $\bar{s} \in S$ is a <u>Nash equilibrium</u> of decision mechanism d if for all $i \in I$, $d(\bar{s})R_i d(\bar{s}/s_i)$ for all $s_i \in S_i$. I.e., $\bar{s} \in S$ is a Nash equilibrium if no individual can unilaterally change his or her strategy and be better off if all other individuals j continue playing $\bar{s}_j \in S_j$.

Let $\bar{s}/s_C$ denote the case where all $i \in C$ choose unbarred strategies, $s_i \in S_i$ and all $j \notin C$ choose barred strategies $\bar{s}_j \in S_j$. Now, $\bar{s} \in S$ is a <u>strong Nash equilibrium</u> of decision mechanism d if there is no $C \subseteq I$ and no $\bar{s}/s_C \in S$ such that $d(\bar{s}/s_C)P_i d(\bar{s})$ for all

$i \in C$.

A social choice correspondence f: $W^I \rightarrow A$ is _implementable_ by decision mechanism d if

(1) for any equilibrium $\bar{s}$ of d with respect to a profile $p \in W^I$, $d(\bar{s}) \in f(p)$;

and (2) if f(p) is nonempty for a profile $p \in W^I$, then d has at least one equilibrium under p.

A social choice correspondence f: $W^I \rightarrow A$ is _fully implementable_ by decision mechanism d if

(1) for any equilibrium, $\bar{s}$, of d with respect to a profile $p \in W^I$, $d(\bar{s}) \in f(p)$;

and (2) for any profile p of individual weak orderings and any $a \in f(p)$ there exists an $\bar{s} \in S$ such that $d(\bar{s}) = a$.

Each of these implementation concepts only makes sense when an equilibrium concept is specified. When the equilibrium concept is Nash equilibrium, then the terminology _fully Nash implementable_ means that the Nash equilibria of the decision mechanism fully implement the social choice process for all profiles in W^I. _Nash implementable_, _fully strong Nash implementable_, and _strong Nash implementable_ are defined similarly.

III. FULL NASH IMPLEMENTATION OF SOCIAL CHOICE CORRESPONDENCES OVER CCE SUBSET ALTERNATIVE SETS

Maskin (1977) and Maskin (1979) provide a set of results for full implementation of social choice correspondences by Nash and strong Nash equilibria when the alternative set is finite. By Theorem

2 in Maskin (1977) Maskin monotonicity is a necessary condition for a social choice correspondence to be fully Nash implementable. A parallel result, Theorem 1 in Maskin (1979) is that Maskin monotonicity is a necessary condition for a social choice correspondence to be fully strong Nash implementable. The proofs of both results do not depend on the alternative set being finite, and thus both results apply when the alternative set is a CCE-subset. The following theorem therefore holds true.

Theorem 1. (Maskin (1977); Maskin (1979)). If a social choice correspondence f over a finite alternative set or over a CCE subset alternative set is fully Nash implementable or fully strong Nash implementable, then f is Maskin monotonic.

Maskin's proof of his Theorem 5 in Maskin (1977) that Maskin monotonicity and NVP are jointly sufficient for a social choice correspondence to be fully Nash implementable does depend on the alternative set being finite. The main task of this section is to show that his result still holds when the alternative set is a CCE subset. Maskin's proof of his Theorem 5 is by construction and rests on his Theorem 4 in the same paper. That theorem and the constructive proof of Theorem 5 utilize a particular strategy space for individuals. In particular, where the social choice correspondence is f, the strategy set for individual i is

$$S_i = \{(R_1,\ldots,R_n,a) \mid p = (R_1,\ldots,R_n) \text{ is a profile of weak orderings and } a \in f(p)\}.$$

Thus, each individual chooses (1) a profile that has a nonempty choice set under f and (2) an element in the choice set for that profile. The profile chosen by i may misrepresent the true preferences of others as well as misrepresenting i's own true preferences. Maskin also defines the lower contour set of R_i at $a \varepsilon A$ as $L(a, R_i) = \{b \varepsilon A | aR_ib\}$.

Theorem 4 in Maskin (1977) is now restated as Theorem 2 here.

<u>Theorem 2</u>. (Maskin (1977)). For $|I| \geq 3$ and a social choice correspondence f that is Maskin monotonic and satisfies NVP, the following three conditions are jointly sufficient for f to be fully Nash implementable by a decision mechanism d: S -> A.

(1) If every individual chooses the same strategy $\bar{s}_i = (R_1,\ldots,R_n,a)$, then the decision mechanism chooses a.

(2) If all individuals except i choose the same strategy $\bar{s}_j = (R_1,\ldots,R_n,a)$ then i can reach any point in $L(a,R_i)$ by some strategy $s_i \varepsilon S_i$. I.e., $\{b \varepsilon A | b = d(\bar{s}_1,\ldots,\bar{s}_2,\ldots,s_i,\ldots,\bar{s}_n)$ and $s_i \varepsilon S_i\} = L(a,R_i)$.

(3) If at least two individuals have different stategies, then any third individual, j, can cause the outcome of the decision mechanism to be <u>any</u> alternative in A by choosing an appropriate $s_j \varepsilon S_j$.

After proving Theorem 2, Maskin proves the joint sufficiency of Maskin monotonicity and NVP for a social choice correspondence to be fully Nash implementable by constructing a decision mechanism for all social choice correspondences that meets (1)-(3) in Theorem 2. As Theorem 3,

this section proves Maskin's joint sufficiency result for finite alternative sets using a different construction than Maskin used. That different construction is then altered so that it applies for CCE subset alternative sets.

<u>Theorem 3</u> (Maskin (1977)). If $|I| \geq 3$, A is finite and a social choice correspondence f satisfies NVP, then f is fully Nash implementable by a decision mechanism if and only if f is Maskin monotonic.

<u>Proof</u>: By Theorem 1 if f is fully Nash implementable, then f is Maskin monotonic.

The following proof that Maskin monotonicity and NVP imply that f is fully Nash implementable is by a construction similar to that in Maskin (1977, pp. 21-22). Specifically, only (B) in the decision mechanism constructed below differs from Maskin's construction.

First, label the alternatives in the alternative set $A = \{a(0),\ldots,a(m-1)\}$. Now consider the following decision mechanism d: S -> A where S_i for each $i \varepsilon A$ is as stated in the paragraph above Theorem 2 <u>supra</u>.

(A) If $s_1 = s_2 = \cdots = s_n = (R_1,\ldots,R_n,a)$, then $d(s_1,\ldots,s_n) = a$.

(B) If $s_1 = s_2 = \cdots = s_{i-1} = s_{i+1} = s_{i+2} = \cdots = s_n = (R_1,\ldots,R_n,a(r)) \neq s_i = (\bar{R}_1,\ldots,\bar{R}_n,a(s))$, $d(s_1,\ldots,s_n) = a(s)$ if $a(r)R_ia(s)$ and $a(r)$ otherwise.

(C) If all individuals pick one of two strategies and more than one picks each of the two strategies or if there exists a triple (i,j,k) such that $s_i \neq s_j \neq s_k \neq s_i$, then let

$d(s_1,\ldots,s_n) = a(\text{mod}_{(m-1)}(\sum_{r=1}^{n} t_r))$ where

$s_i = (R_1^i,\ldots,R_n^i,a(t_i))$ and where $\text{mod}_x(y) = k$ when $y = ax + k$ with ax the largest multiple of x less than or equal to y.

This construction completely defines d. The decision mechanism d satisfies (1) of Theorem 2 because of (A). Now by (B) any individual i who deviates from $(R_1,\ldots,R_n,a)$, the strategy common to the other individuals, can obtain any alternative in the lower contour set of R_i at a. Thus, the construction meets (2) of Theorem 2.

When any two individuals have different strategies, then for each $a(v) \varepsilon A$ any third individual i can select a strategy s_i distinct from each of the first two and s_i will consist of a profile p and an alternative a(v) such that $a(v) \varepsilon f(p)$. (Note that by NVP, there are at least n + 1 profiles that will result in any $a \varepsilon A$ being in the choice set for any social choice correspondence f. Specifically, all the profiles that have a at the top of the weak orderings of at least n − 1 individuals will result in a being in the choice set, and with two or more alternatives there are at least n + 1 such profiles.) Thus individual i can cause the sum $\sum_{r=1}^{n} t_r$ in (C) to be any of m consecutive integers since i can select an s_i such that $t_i = 0$, $t_i = 1,\ldots,$ or $t_i = m - 1$. It follows from (C) that i can cause any alternative in A to be the outcome under the decision mechanism by choosing an appropriate s_i. Thus, condition (3) of Theorem 2 is met.

Q.E.D.

In order to extend the construction in the proof of Theorem 3 to the case of a CCE subset alternative set, the most important task is to revise condition (C). Condition (C) depends on the ability to label n alternatives by positive integers starting with one and ending with n. The following theorem extends the result of Theorem 3 to CCE subset alternative sets by altering condition (C) in the proof without affecting the applicability of Theorem 2.

Theorem 4: If each individual weak ordering can have any alternative at the top of the ordering, if $|I| \geq 3$, if A is a CCE subset and if a social choice correspondence f satisfies NVP, then f is fully Nash implementable by a decision mechanism if and only if f is Maskin monotonic.

Proof: By Theorem 1 if f is fully Nash implementable, then f is Maskin monotonic.

In order to show that Maskin monotonicity and NVP imply that f is fully Nash implementable, a decision mechanism satisfying (A) and (B) in the proof of Theorem 3 can be used, but condition (C) in that proof must be altered. This proof develops a new condition (C′) to replace (C) and then demonstrates that (A), (B) and (C′) generate a decision mechanism that meets the conditions of Theorem 2.

Suppose A has dimension m. I.e., A is an m-CCE subset. A is embedded in $\mathbb{R}^m$, and since A is compact and convex, for each point $x \in \mathbb{R}^m$ not in A there is a point $y \in A$ such that y is the point in A closest to x if $x \notin A$. Define a function $Q : \mathbb{R}^m \to A$ such that $Q(x) = x$ if $x \in A$ and $Q(x) = y$ where y is the point in A closest to x if $x \notin A$. Let each alternative $a \in A$ be a vector characterized by m coordinates in the coordinate system of $\mathbb{R}^m \supset A$.

For a social choice correspondence f let the strategy space for each individual i be

$$S_i' = \{(R_1,\ldots,R_n,a,b) \mid b \in \mathbb{R}^m \text{ and } a \in f(R_1,\ldots,R_n)\}.$$

Denote as s_i', the ith individual's choice from S_i', and let s_i be the vector of all components of s_i' but the last. Let the last two components of s_i' be denoted a_i and b_i respectively.

Define condition (C′) as follows:

(C′) If (1) the set $T = (s_1,\ldots,s_n)$ consists of two subsets T_1 and T_2 such that $T_1 + T_2 = T$, $|T_1| \geq 2$, $|T_2| \geq 2$, and for s_i, $s_j \in T_1$ and s_k, $s_r \in T_2$ it is true that $s_i = s_j$, $s_k = s_r$ and $s_i \neq s_k$ or (2) there exists a triple (i,j,k) such that $s_i \neq s_j \neq s_k \neq s_i$, then let $d(s_1',\ldots,s_n') = Q(\sum_{w=1}^{n} b_w)$.

This condition along with conditions (A) and (B) from the proof of Theorem 3 completely define a decision mechanism d: $S' \to A$ where S' is the product space $\prod_{i=1}^{n} S_i'$. Note that conditions (A) and (B) operate only on the portion s_i of each s_i'. I.e., b_i, the last vector component of s_i', is ignored in applying (A) and (B). For example, if $s_i' = (R_1,\ldots,R_n,a,b_i)$ and $b_j \neq b_k$ for some j, $k \in I$, then condition

(A) will still be satisfied and $d(s_1',\ldots,s_n') = a$. Note also that although condition (C′) makes reference to the b_i components, in determining whether condition (C′) applies instead of (A) or (B) reference is made only to the s_i portion of s_i'.

The proof of Theorem 2 as Theorem 4 in Maskin (1977) relies on three properties: $a \in f(R_1,\ldots,R_n)$, f is Maskin monotonic, and f satisfies NVP. The proof is unaffected by shifting the individual strategy space for each individual i from S_i to S_i' as long as the applicability of conditions (1), (2) and (3) depend only on the s_i portion of each vector s_i' since that portion is a member of S_i. In other words, adding an extra vector component b_i to each $s_i \in S_i$ makes no operational difference in the proof if (1), (2) and (3) do not involve that extra vector component.

It is important to check, then, that (1), (2), and (3) can be made to depend only on the s_i portion of the s_i' vectors. There is no problem with (1) and (2). Both of these conditions can be interpreted in terms of the s_i portion of the s_i' vectors. So when either (1) or (2) refer to s_i or $\bar{s}_i$ these should be understood to mean the s_i and $\bar{s}_i$ portions of s_i' and $\bar{s}_i'$ respectively.

There is, however, one subtlety in interpreting the conditions of Theorem 2 when the individual strategy sets are S_i' instead of S_i. Condition (3) of Theorem 2 applies "if at least two individuals have different strategies." This is to be interpreted as "if at least two individuals h and i have $s_h \neq s_i$." In other words, the individuals need only have strategies that differ on the s_k portion of the s_k'

vectors. If it is true that $s_h = s_i$ for all h, i ε I, then (3) does not apply. On the other hand, if (3) does apply the result will be that any third individual j can cause the outcome of the decision mechanism to be <u>any</u> alternative in A by choosing an appropriate $s_j' \varepsilon S_j'$. The b_j component in s_j' matters for this choice.

So the only place where (1), (2), or (3) in Theorem 2 depend on the b_i portion of the s_i' is in part of (3). But that dependence does not matter. The condition (3) specifies that when it applies, a third individual j can force the decision mechanism to select any outcome in A by appropriate choice of a strategy vector. The condition and the proof of the theorem are not concerned with how this power in the third individual arises. In particular, that power can come about through an extra component in the strategy vector. For the purposes of Theorem 2 it only matters that the power exists.

The three properties (A), (B), and (C′) completely specify a decision mechanism d. Now that it is clear that changing the individual strategy sets to S_i' has no effect on the operation of Theorem 2 if conditions (1), (2) and (3) in that theorem are interpreted properly, it remains to show that (A), (B), and (C′) satisfy these conditions in their properly interpreted form.

The decision mechanism satisfies condition (1) of Theorem 2 by (A). As long as any individual can choose a weak ordering that puts any alternative at the top of the ordering, then by NVP and (B), condition (2) of Theorem 2 is satisfied. Finally by (C′) when any two individuals i and j have $s_i \neq s_j$ any third individual k can make $\sum_{w=1}^{n} b_w$

be any vector in $\mathbb{R}^m$ given that all b_i for i $\neq$ k remain unchanged. Thus, by choosing an appropriate $s_k' \varepsilon S_k'$ k can cause the game to produce any alternative in the alternative set.

Q.E.D.

The condition in Theorem 4 that any individual be able to choose an individual weak ordering that puts any alternative t ε A at the top of that weak ordering is not very restrictive. For example, given that type I preferences are admissible for individual i, i can simply choose bliss point t to insure that t will be at the top of i's weak ordering.

IV. CHARACTERIZING FULLY IMPLEMENTABLE NEUTRAL SOCIAL FUNCTIONS

The first task in this section is to show that neutral social functions that are Maskin monotonic are neutral monotonic social functions. In the beginning that task, the following lemma is useful:

<u>Lemma 1</u>. A neutral social function is characterized by a collection of proper games:

(a) If Γ is a collection of proper games then μ_Γ, the aggregation rule generated by Γ, is a neutral social function.

(b) If σ is a neutral social function, then there exists a unique collection of proper games, Γ, such that $\sigma = \mu_\Gamma$, the aggregation rule generated by Γ.

Proof: (a) Suppose Γ is a collection of proper games. Suppose that $a\mu_\Gamma b$ for some a, b ε A and some profile p ε W^I. By the definition of μ_Γ it follows that $p(a > b) \varepsilon \Gamma_J$ where $J = p(a > b) \cup p(b > a)$. But since Γ_J is a proper game, $p(b > a) \notin \Gamma_J^*$ and thus it is not true that $b\mu_\Gamma a$. Since $a\mu_\Gamma b \Rightarrow b\mu_\Gamma a$ is not true, μ_Γ is a social function. Since μ_Γ is generated by a proper game, which coalitions win or lose is independent of the particular pair of alternatives being compared. As a result, μ_Γ is neutral.

(b) If σ is a neutral social function, σ is a neutral binary decision rule since neutrality implies binaryness. Ferejohn and Fishburn (1979) show as their Theorem 1 that for any binary decision rule, the set of winning coalitions for any concerned set and any pair of alternatives is a "binary constitution." A binary constitution guarantees that for any pair of alternatives x, y ε A, if coalition W is a winning coalition then its complement L in the concerned set will be a losing coalition. When there is a binary constitution, the collection of winning coalitions for x versus y for a given concerned set satisfies the criterion of a proper game. In particular, if a given coalition wins in a proper game, then its complement in the concerned set does not win. But a binary constitution structure does not yield a proper game for each concerned set. The reason is that different coalitions may win for a given concerned set depending on which two alternatives are being compared. This possibility is eliminated by assuming neutrality since neutrality guarantees that the set of winning coalitions for a given concerned set does not depend on

which two alternatives are being compared. Thus, Ferejohn and Fishburn's Theorem 1 and the additional assumption of neutrality yield the result that for any neutral binary decision rule, the set of winning coalitions for each concerned set is a proper game.

Now suppose that there are two distinct collections of proper games, Γ and $\tilde{\Gamma}$ and that $\sigma = \mu_\Gamma$ and $\sigma = \mu_{\tilde{\Gamma}}$ where σ is a neutral binary decision rule. Since Γ and $\tilde{\Gamma}$ are distinct assume (without loss of generality) that for some $J \subset I$ there exists a coalition E such that E ε $\tilde{\Gamma}_J/\Gamma_J$. Consider two alternatives x, y ε A. Choose a profile p ε W^I such that $p(x > y) = E$, $p(y > x) = J/E$. (Such a profile can be easily constructed even when A is a CCE subset if type I preferences are admissible for all individuals. E.g., let members of E have bliss point x, members of J/E have bliss point y and members of I/J have a bliss point equidistant from x and y.) Now since E ε $\tilde{\Gamma}_J$ and $\mu_{\tilde{\Gamma}} = \sigma$, $x\sigma y$. But E $\notin \Gamma_J$ and $\mu_\Gamma = \sigma$ imply that $x\sigma y$ is not true, a contradiction. Thus, Γ, the collection of proper games such that $\mu_\Gamma = \sigma$, must be unique.

Q.E.D.

Given that the set of winning coalitions for each concerned set is a proper game, the next step is to show that all the proper games are simple games. The following lemma accomplishes that goal. The proof of that lemma as well as the four succeeding lemmas is by counterexample. It is important to be sure that the counterexamples are acceptable even under the most severe restrictions on preferences used in the rest of the article. To that end, the proof of each lemma is accompanied by a demonstration that the particular profiles used as counterexamples in the proof are consistent with profiles of weak orderings in the finite alternative set case and with profiles of weak orderings that admit type I preferences when the alternative set is a CCE subset.

Lemma 2: If a neutral social function is Maskin monotonic, then any proper game representing a set of winning coalitions for a given concerned set is a simple game.

Proof: Call the neutral social function f so that its choice set under profile r is f(r). To show that a proper game representing the set of winning coalitions under f for a given concerned set is a simple game, it is only necessary to show that it is a monotonic game. For a concerned set $H \subseteq I$ label as Γ_H the proper game associated with that concerned set. Assume that a proper game, Γ_J, with respect to some concerned set $J \subseteq I$ is not a monotonic game. Then either

(1) there exists a coalition $C \in \Gamma_J$ such that for some $D \supset C$, $J/D \in \Gamma_J$,

or (2) there exists a coalition $E \in \Gamma_J$ such that for some $G \supset E$, $G \notin \Gamma_J$ and $J/G \notin \Gamma_J$.

The first part of this proof will show that (1) and (2) contradict Maskin monotonicity by using p, q, u and v, four particular

profiles of weak orderings, in the case where the alternative set is finite. Let the profiles used have the following traits where a, $b \in A$: (1') J is the concerned set for a versus b; (2') for all $i \in J$, aP_ic and bP_ic for all $c \in A$ where $c \notin \{a, b\}$.

Under a profile of this type, note that for $x \in \{a,b\}$ and $y \in A/\{a,b\}$ $I = \{i \in I | xP_iy\}$. As a first step, it is important to verify that $\emptyset \notin \Gamma_I$. Under that condition, there will be no $c \in A/\{a,b\}$ such that cPd where $d \in \{a,b\}$. In other words, a will be in the choice set unless bPa, and b will be in the choice set unless aPb.

Consider two profiles r and s. Under profile r for all $i \in I$ and all $x, y \in A$, xI_iy. Under profile s for all $i \in I$ and for all $x \in A/\{z\}$ zP_ix, but for all $i \in I$ and for all $x, y \in A/\{z\}$ xI_iy. Under profile r no one is concerned about any binary choice so that the concerned set is $\emptyset$. Since $\Gamma_\emptyset$ is a proper game and since $\emptyset$ is the complement of itself in $\Gamma_\emptyset$, it must be true that $\emptyset \notin \Gamma_\emptyset$. As a result, under r there is social indifference between all alternatives, and all alternatives are in the choice set. Now suppose $\emptyset \in \Gamma_I$. Then under profile s, $z \notin f(s)$ since xPz for all $x \in A/\{z\}$. The transformation r -> s is such that z is not demoted in any individual's weak ordering. Maskin monotonicity requires that $z \in f(r) \Rightarrow z \in f(s)$ so that $z \notin f(s)$ is a contradiction. Thus, $\emptyset \notin \Gamma_I$.

Now consider case (1). Under a profile p let $C = \{i | aP_ib\}$. Since $C \in \Gamma_J$, $a \in f(p)$. Change the profile such that the new profile q has $D = \{i | aP_ib\}$. Since $J/D \in \Gamma_J$, bPa and $a \notin f(q)$. The transformation from p to q does not demote a in anyone's weak ordering since the set $\{i|bP_i a\}$ does not expand under the transformation. Yet $a \notin f(q)$ which contradicts Maskin monotonicity.

Similar reasoning shows that (2) leads to a contradiction of Maskin monotonicity. Let $G = \{i|bP_i a\}$ under profile u, and let $E = \{i|bP_i a\}$ under profile v. Now $a \in f(u)$ but $a \notin f(v)$. But the transformation from u to v does not demote a in anyone's weak ordering since the set $\{i|bP_i a\}$ does not expand under the transformation. Thus, $a \in f(u)$ and $a \notin f(v)$ contradicts Maskin monotonicity.

The first part of the proof is now complete: (1) and (2) each lead to a contradiction of Maskin monotonicity for the finite alternative set case when individual preferences can be any weak ordering.

The second part of the proof presents type I preference profiles over CCE subset alternative sets that cause each of (1) and (2) to lead to a contradiction of Maskin monotonicity. Choose x, $y \in A$ where A is a CCE subset. In case (1) set up two profiles. Under profile p all members of C have type I preferences with x as a bliss point, members of J/C have type I preferences with y as a bliss point, and members of I/J are indifferent between any two alternatives in A. Under profile q members of D have type I preferences with x as a bliss point, members of J/D have type I preferences with y as a bliss point, and members of I/J are indifferent between any two alternatives in A.

The following arguments establish that $x \in f(p)$. For

$z \in A/\{x\}$ there are three cases:

(A) $|z - y| < |x - y|$ where $|\cdot|$ is Euclidean distance;

(B) $|z - y| > |x - y|$;

(C) $|z - y| = |x - y|$.

Under p in case (A), $C = \{i|xP_i z\}$ and $J/C = \{i|zP_i x\}$. Since $C \in \Gamma_J$, xPz for all $z \in A$ where (A) holds.

In case (B), $J = \{i|xP_i z\}$ and $\emptyset = \{i|zP_i x\}$. Consider two profiles g and h. Under profile g for all $i \in I$ and all a, $b \in A$, $aI_i b$. Under profile h all $i \in J$ have type I preferences with bliss point a and for all $i \in I/J$ and all b, $c \in A$, $bI_i c$. Under profile g no one is concerned about any binary choice so that the concerned set is $\emptyset$. Since $\Gamma_\emptyset$ is a proper game and since $\emptyset$ is the complement of itself in $\emptyset$ it must be true that $\emptyset \notin \Gamma_\emptyset$. As a result, under g there is social indifference between all alternatives, and all alternatives are in the choice set. Now suppose $\emptyset \in \Gamma_J$ for $J \subseteq I$. Then under profile h, $a \notin f(h)$ since bPa for all $b \in A/\{a\}$. The transformation $g \rightarrow h$ is such that a is not demoted in any individual's weak ordering. Maskin monotonicity requires $a \in f(g) \Rightarrow a \in f(h)$ so that $a \notin f(h)$ is a contradiction. Thus, $\emptyset \notin \Gamma_J$. For all $z \in A$ for which case (B) applies, xRz.

In case (C), $C = \{i|xP_i z\}$ and $\emptyset = \{i|zP_i x\}$. By an argument similar to that for case (B), $\emptyset \notin \Gamma_C$. Thus, whenever case (C) holds for $z \in A$, xRz.

Combining the analysis of the three cases, it is true that xRz where any of (A), (B) or (C) hold. But those cases are exhaustive so that $x \in f(p)$.

Under profile q consider the binary choice between x and y. The concerned set is J and $J/D = \{i|yP_ix\}$. Since $J/D \in \Gamma_J$, yPx and $x \notin f(q)$. Since the only change in individual profiles in the transformation $p \rightarrow q$ is in the group D/C and since members in that group switch their bliss point from y to x under the transformation, x is not demoted in any individual's weak ordering by the transformation. By Maskin monotonicity it must be true that $x \in f(p) \Rightarrow x \in f(q)$. Thus, $x \notin f(q)$ violates Maskin monotonicity.

Now consider the following profiles u and v for case (2). Under profile u, E is a set of individuals who have type I preferences with bliss point y, individuals in J/E have type I preferences with bliss point x, and individuals in I/J are indifferent between any two alternatives in A. Under profile v, G is a set of individuals who have type I preferences with bliss point y, individuals in J/G have type I preferences with bliss point x, and individuals in I/J are indifferent between any two alternatives in A.

The following arguments establish that $x \in f(v)$. For $z \in A/\{x\}$ there are three cases:

(A) $|z - y| < |x - y|$ where $|\cdot|$ is Euclidean distance;

(B) $|z - y| > |x - y|$;

(C) $|z - y| = |x - y|$.

Under v in case (A), $G = \{i|zP_ix\}$ and $J/G = \{i|xP_iz\}$. Since $G \notin \Gamma_J$, xRz for all $z \in A$ where (A) holds.

In case (B), $J = \{i|xP_iz\}$ and $\emptyset = \{i|zP_ix\}$. Since case (1) is

ruled out by Maskin monotonicity, $E \in \Gamma_J \Rightarrow \emptyset \notin \Gamma_J$ so that J cannot be a losing coalition for concerned set J. It follows that xRz for all $z \in A$ where (B) holds.

In case (C), $J/G = \{i|xP_iz\}$ and $\emptyset = \{i|zP_ix\}$. Since case (1) is ruled out by Maskin monotonicity and since $\Gamma_{J/G}$ is a proper game, $\emptyset \in \Gamma_{J/G}$ cannot be true. ($\emptyset \in \Gamma_{J/G} \Rightarrow J/G \in \Gamma^*_{J/G}$ since $\Gamma_{J/G}$ is a proper game. But $\emptyset \in \Gamma_{J/G}$ and $J/G \in \Gamma^*_{J/G}$ falls within case (1), and case (1) has been shown to be impossible.) Thus, either $J/G \in \Gamma_{J/G}$ or $J/G \notin \Gamma_{J/G}$ and $\emptyset \notin \Gamma_{J/G}$. It follows that xRz for all $z \in A$ where (C) holds.

Combining the analysis of the three cases, it is true that xRz where any of (A), (B) or (C) hold. But those cases are exhaustive, so $x \in f(v)$.

The transformation $v \rightarrow u$ involves changing the preferences of only one group, G/E. These individuals switch their bliss point from y to x so that x is not demoted in any individual's weak ordering by the transformation. Maskin monotonicity requires that $x \in f(v) \Rightarrow x \in f(u)$. Under profile u, $E = \{i|yP_ix\}$ and $J/E = \{i|xP_iy\}$. $E \in \Gamma_J \Rightarrow$ yPx, and therefore $x \notin f(u)$.

Q.E.D.

Lemma 2 states that a Maskin monotonic neutral social function is characterized by a simple game for each concerned set. Maskin monotonicity further implies that these simple games form a direct sum of simple games. Before proving that result as Lemma 5, it is convenient to develop two intermediate results as Lemma 3 and Lemma 4.

<u>Lemma 3</u>. If a neutral social function is Maskin monotonic, and for some $C \subseteq J \subseteq I$ it is true that $C \varepsilon \Gamma_J$, the simple game for concerned set J, then $J/C \notin \Gamma_K$ for any $K \subseteq I$ where Γ_K is the simple game for concerned set K.

<u>Proof</u>: Call the neutral social function f so that f(r) is the choice set of f for profile r. Denote as Γ_K the simple game corresponding to f for concerned set K.

The first part of the proof establishes Lemma 3 when A is a finite alternative set and individual preferences are weak orderings. Two profiles p and q of weak orderings over A will be used, and they will both have the property that for $c \varepsilon A/\{a,b\}$, aP_ic and bP_ic for all $i \varepsilon I$. Since Γ_I is a simple game, $\emptyset \notin \Gamma_I$. (If $\emptyset \varepsilon \Gamma_I$, then $I \notin \Gamma_I$ since Γ_I is a proper game. But $\emptyset \varepsilon \Gamma_I \Rightarrow I \varepsilon \Gamma_I$ since Γ_I is a monotonic game.) As a result, a will be in the choice set unless bPa, and b will be in the choice set unless aPb.

Under profile p let $C = \{i|aP_ib\}$ and $J/C = \{i|bP_ia\}$. Then a is in the choice set for f(p). Suppose $J/C \varepsilon \Gamma_K$ for some $K \subseteq I$. Then the profile q where $J/C = \{i|bP_ia\}$ and the concerned set for a versus b is K has the property that $a \notin f(q)$ since J/C is a winning coalition. But the transformation p -> q does not demote a in any individual's weak ordering since it does not expand the set $\{i|bP_ia\}$. Thus, $J/C \varepsilon \Gamma_K$ violates the assumption that f is Maskin monotonic.

Now as the second part of the proof, it is necessary to establish Lemma 3 when A is a CCE subset and type I preferences are

admissible. This part of the proof operates by presenting profiles such that $J/C \varepsilon \Gamma_K$ contradicts Maskin monotonicity. Suppose $x, y \varepsilon A$. Under profile u members of C have type I preferences with bliss point x, members of J/C have type I preferences with bliss point y, and members of I/J are indifferent between any two alternatives in A. Under profile v members of J/C have type I preferences with bliss point y, members of K/(J/C) have type I preferences with bliss point x, and members of I/K are indifferent between any two alternatives in A.

The following arguments establish that $x \varepsilon f(u)$. For $a \varepsilon A/\{x\}$ there are three cases:

(1) $|z - y| < |x - y|$ where $|\cdot|$ is Euclidean distance;

(2) $|z - y| > |x - y|$;

(3) $|z - y| = |x - y|$.

In case (1) $C = \{i|xP_iz\}$ and $J/C = \{i|zP_ix\}$. Since $C \varepsilon \Gamma_J$, xPz for $z \varepsilon A$ where case (1) applies. In case (2), $J = \{i|xP_iz\}$ and $\emptyset = \{i|zP_ix\}$. Since Γ_J is a simple game, $\emptyset \notin \Gamma_J$ so that xRz for all $z \varepsilon A$ where case (2) applies. In case (3), $C = \{i|xP_iz\}$ and $\emptyset = \{i|zP_ix\}$. Since Γ_C is a simple game, $\emptyset \notin \Gamma_C$ so that xRz for all $z \varepsilon A$ where case (3) applies. Since xRz for $z \varepsilon A$ in cases (1), (2) and (3) and since these cases exhaust the possibilities, $x \varepsilon f(u)$.

Under profile v, K is the concerned set for x versus y and $J/C = \{i|yP_ix\}$. It follows that yPx and $x \notin f(v)$. The transformation u -> v does not demote x in any individual's weak ordering because the set of individuals with bliss point y does not expand. All other individuals, those in $I/(J/C)$, either have bliss point x or are universally indifferent under each of the profiles. For $i \in I/(J/C)$ it is true that xR_iz for all $z \in A$ under either profile. Since the transformation $u \rightarrow v$ does not demote x in any individual's weak ordering, Maskin monotonicity requires that $x \in f(u) \Rightarrow x \in f(v)$. This is contradicted by the fact that $x \in f(u)$ and $x \notin f(v)$.

Q.E.D.

Lemma 3 states that if a coalition loses in the simple game for any concerned set, it cannot be a winning coalition in the simple game for any other concerned set. Similar reasoning to the proof of Lemma 3 produces the next result: if a coalition neither wins nor loses in the simple game for any concerned set, it cannot be a winning coalition in the simple game for any other concerned set.

<u>Lemma 4</u>. If a neutral social function is Maskin monotonic and for some $C \subseteq J \subseteq I$ it is true that $C \notin \Gamma_J$ and $J/C \notin \Gamma_J$ where Γ_J is the simple game for concerned set J, then $C \notin \Gamma_K$ for any $K \subseteq I$ where Γ_K is the simple game for concerned set K.

<u>Proof</u>: Call the neutral social function f so that its choice set for profile r is f(r). Let Γ_K be the simple game corresponding to f for concerned set K.

The first part of this proof establishes Lemma 4 when A is a finite alternative set and individual preferences are weak orderings. Two profiles, p and q, of weak orderings over A will be used, and both have the property that for $c \in A/\{a,b\}$ aP_ic and bP_ic for all $i \in I$.

Since Γ_I is a simple game, $\emptyset \notin \Gamma_I$. As a result, a will be in the choice set unless bPa, and b will be in the choice set unless aPb.

Suppose that $C \in \Gamma_K$ for some $K \subseteq I$ when $C \notin \Gamma_J$ and $J/C \notin \Gamma_J$ for some $J \subseteq I$. Let profile p have concerned set J for a versus b, and let profile q have concerned set K for a versus b. Under both profiles let $C = \{i | bP_ia\}$. Then $a \in f(p)$ since $C \notin \Gamma_J$ so that it is not true that bPa. Also, $a \notin f(q)$ since $C \in \Gamma_K$. The transformation $p \rightarrow q$ does not demote a in any individual's weak ordering since the set $C = \{i | bP_ia\}$ does not expand under the transformation and aP_ic for all $c \in A/\{a,b\}$ and for all $i \in I$ under both p and q. Under these circumstances, Maskin monotonicity requires that $a \in f(p) \Rightarrow a \in f(q)$. But $a \in f(p)$ and $a \notin f(q)$ so that there is a contradiction.

The task of the second part of the proof is to establish Lemma 4 when A is a CCE subset and type I preferences are admissible. This part of the proof will present profiles such that the hypothesis $C \in \Gamma_K$ while $C \notin \Gamma_J$ and $J/C \notin \Gamma_J$ contradicts Maskin monotonicity.

Suppose $x, y \in A$. Under profile u, members of C have type I preferences with bliss point y, members of J/C have type I preferences with bliss point x, and members of I/J are indifferent between any two alternatives in A. Under profile v members of C have type I preferences with bliss point y, members of K/C have type I preferences with bliss point x, and members of I/K are indifferent between any two alternatives in A.

The following arguments establish that $x \in f(u)$. For $z \in A/\{x\}$ there are three cases:

(1) $|z - y| < |x - y|$ where $|\cdot|$ is Euclidean distance;

(2) $|z - y| > |x - y|$;

(3) $|z - y| = |x - y|$.

In case (1) $C = \{i|zP_ix\}$ and $J/C = \{i|xP_iz\}$. Since $C \notin \Gamma_J$, xRz for $z \in A$ where case (1) applies. In case (2), $J = \{i|xP_iz\}$ and $\emptyset = \{i|zP_ix\}$. Since Γ_J is a simple game, $\emptyset \notin \Gamma_J$, and consequently xRz for $z \in A$ where case (2) applies. In case (3) $J/C = \{i|xP_iz\}$ and $\emptyset = \{i|zP_ix\}$. Since $\Gamma_{J/C}$ is a simple game, $\emptyset \notin \Gamma_{J/C}$ and consequently xRz for $z \in A$ where case (3) applies. Since xRz for $z \in A$ in cases (1), (2) and (3) and since these cases exhaust the possibilities, $x \in f(u)$.

Under profile v, K is the concerned set for x versus y and $C = \{i|yP_ix\}$. Since $C \in \Gamma_K$, yPx, and therefore $x \notin f(v)$. The transformation u -> v does not demote x in any individual's weak ordering. The set of individuals with bliss point y does not expand. All other individuals, those in I/C, either have bliss point x or are universally indifferent under each of the profiles. Thus, for $i \in I/C$ it is true that xR_iz for all $z \in A$ under either profile u or profile v. Since the transformation u -> v does not demote x in any individual's weak ordering, Maskin monotonicity requires that $x \in f(u) \Rightarrow x \in f(v)$. This is contradicted by the fact that $x \in f(u)$ and $x \notin f(v)$.

Q.E.D.

Lemma 5 builds on Lemmas 2, 3 and 4 to show that if a neutral social function is Maskin monotonic, it is generated by a direct sum

of simple games. By Theorem 1 in Blau and Brown (1978) and by Theorem 1 in Strnad (1982) neutral monotonic social functions are characterized by direct sums of simple games. As a result, a Maskin monotonic neutral social function is a neutral monotonic social function.

Lemma 5. If a neutral social function f is Maskin monotonic, then f is generated by a direct sum of simple games.

Proof: Suppose that for the neutral binary decision rule f and for $K \subseteq I$, Γ_K is the simple game corresponding to f when the concerned set is K. In order to show that $\Gamma = \{\Gamma_K\}_{K \in 2^I}$ is a direct sum of simple games, two characteristics must be established:

(A) $C \in \Gamma_J \Rightarrow C \in \Gamma_K$ where $C \subseteq K \subseteq J$;

(B) $D \in \Gamma^*_J \Rightarrow D \in \Gamma^*_L$ where $J \subseteq L$.

To prove (A), assume that $C \notin \Gamma_K$. Then there are two cases: $K/C \in \Gamma_K$ and $K/C \notin \Gamma_K$. The first case is ruled out by Lemma 3. The second case is ruled out by Lemma 4 because if $C \notin \Gamma_K$ and $K/C \notin \Gamma_K$, then $C \in \Gamma_J$ is impossible under that lemma if f is Maskin monotonic.

Now consider proposition (B) and suppose that $D \notin \Gamma^*_L$. Then there are two cases: (1) $D \in \Gamma_L$, (2) $D \notin \Gamma_L$ and $L/D \notin \Gamma_L$. The first case is inconsistent with Lemma 3 since $D \in \Gamma^*_J$ implies that $D \in \Gamma_L$ cannot be true for any $L \subseteq I$ if f is Maskin monotonic.

For the second case it is necessary to prove the result by constructing profiles that lead to a contradiction of Maskin monotonicity. First, consider the situation where A is a finite set and individual preferences must be weak orderings. Let profiles p and q of weak orderings have the following properties:

(1') for all $c \in A/\{a,b\}$ aP_ic and bP_ic for all $i \in I$ under both profiles;

(2') for a versus b the concerned set under profile p is L and $D = \{i|aP_ib\}$;

(3') for a versus b the concerned set under profile q is J and $D = \{i|aP_ib\}$.

Since $\emptyset \notin \Gamma_I$, a is in the choice set for either p or q unless bPa, and b is in the choice set for either p or q unless aPb. Since $L/D \notin \Gamma_L$, aRb under profile p and $a \in f(p)$. Since $D \in \Gamma^*_J$, bPa under profile q and $a \notin f(q)$. The transformation p -> q does not demote a in any individual's weak ordering because the set of those who prefer b to a contracts from L/D to J/D under that transformation. Under those circumstances Maskin monotonicity requires that $a \in f(p) \Rightarrow a \in f(q)$, but here $a \in f(p)$ and $a \notin f(q)$.

Now consider the situation where A is a CCE subset and type I preferences are admissible. Let $x, y \in A$. Define the profiles u and v as follows. Under profile u, members of D have type I preferences with bliss point x, members of L/D have type I preferences with bliss point y, and members of I/L are indifferent between any two alternatives in A. Under profile v, members of D have type I preferences with bliss point x, members of J/D have type I preferences with bliss point y, and members of I/J are indifferent between any two alternatives in A.

The following arguments establish that $x \in f(u)$. For $z \in A/\{x\}$ there are three cases:

(1) $|z - y| < |x - y|$ where $|\cdot|$ is Euclidean distance;

(2) $|z - y| > |x - y|$;

(3) $|z - y| = |x - y|$.

In case (1) $D = \{i|xP_iz\}$ and $L/D = \{i|zP_ix\}$. Since $L/D \notin \Gamma_L$, xRz for all $z \in A$ where case (1) holds. In case (2) $L = \{i|xP_iz\}$ and $\emptyset = \{i|zP_ix\}$. Since Γ_L is a simple game, $\emptyset \notin \Gamma_L$, and xRz for all $z \in A$ where case (2) holds. In case (3) $D = \{i|xP_iz\}$ and $\emptyset = \{i|zP_ix\}$. Since Γ_D is a simple game, $\emptyset \notin \Gamma_D$, and xRz for all $z \in A$ where case (3) holds. Since xRz for $z \in A$ in cases (1), (2) and (3) and since these cases exhaust the possibilities, $x \in f(u)$.

Under profile v, J is the concerned set for x versus y and $D = \{i|xP_iy\}$. Since $D \in \Gamma^*_J$, yPx, and therefore $x \notin f(v)$. The transformation u -> v does not demote x in any individual's weak ordering. The set of individuals with bliss point y contracts from L/D to J/D. For all $i \in I/(J/D)$ xR_iz for all $z \in A$ under profile v. Since the transformation u -> v does not demote x in any individual's weak ordering, Maskin monotonicity requires that $x \in f(u) \Rightarrow x \in f(v)$. This is contradicted by the fact that $x \in f(u)$ and $x \notin f(v)$.

Q.E.D.

Lemma 5 shows that Maskin monotonicity limits neutral social functions to the class of neutral monotonic social functions. The following lemma establishes that Maskin monotonicity leads to further restrictions.

Lemma 6. If a neutral social function is Maskin monotonic, then it is a neutral monotonic social function that is simple.

Proof: By Lemma 5, a Maskin monotonic neutral social function is generated by a direct sum of simple games. By Theorem 1 in Blau and Brown (1978), and Theorem 1 in Strnad (1982), f is therefore a neutral monotonic social function.

Suppose that Γ is the direct sum of simple games that generates f. Using Lemmas 3 and 4 it is straightforward to show that f is simple if it is Maskin monotonic. Lemma 3 provides that if f is Maskin monotonic, then for any $C \subseteq I$ if there is a $J \subseteq I$ such that $C \varepsilon \Gamma^*_J$, then $C \not\varepsilon \Gamma_K$ for any K such that $C \subseteq K \subseteq I$. It follows that if $C \varepsilon \Gamma_H$ for some $H \subseteq I$, then $C \not\varepsilon \Gamma^*_L$ for any L such that $C \subseteq L \subseteq I$. Lemma 4 provides that if f is Maskin monotonic, then for any $C \subseteq I$ if there is a $J \subseteq I$ such that $C \not\varepsilon \Gamma_J$ and $J/C \not\varepsilon \Gamma_J$ then $C \not\varepsilon \Gamma_K$ for any K such that $C \subseteq K \subseteq I$. It follows that if $C \varepsilon \Gamma_H$ for some $H \subseteq I$, then there will be no $L \subseteq I$ where $C \subseteq L$, $C \not\varepsilon \Gamma_L$ and $L/C \not\varepsilon \Gamma_L$.

Collecting these results, if f is Maskin monotonic and for any $C \subseteq I$ if $C \varepsilon \Gamma_H$ for some $H \subseteq I$ then

(1) for all J such that $C \subseteq J \subseteq I$ $C \not\varepsilon \Gamma^*_J$;

and (2) for all K such that $C \subseteq K \subseteq I$ it is never true that $C \not\varepsilon \Gamma_K$ and $K/C \not\varepsilon \Gamma_K$.

From (1) and (2) if f is Maskin monotonic and $C \varepsilon \Gamma_H$ for some $H \subseteq I$ then for all J such that $C \subseteq J \subseteq I$, $C \varepsilon \Gamma_J$. Thus if f is Maskin monotonic, $C \varepsilon \Gamma_H$ for some $H \subseteq I$ if and only if $C \varepsilon \Gamma_I$. So for any x,

$y \varepsilon A$ xPy if and only if $\{i|xP_iy\} \varepsilon \Gamma_I$. It follows that f is simple.
Q.E.D.

Given Lemma 6, it is possible to prove the following characterization theorem.

Theorem 5. A neutral social function is Maskin monotonic if and only if it is a neutral monotonic social function that is simple.

Proof: Maskin monotonicity is a sufficient condition for f to be a neutral monotonic social function that is simple by Lemma 6.

Suppose f is a simple neutral monotonic social function. Then by Theorem 1 in Blau and Brown (1978) and Theorem 1 in Strnad (1982) there exists a unique direct sum of simple games, Γ, such that $f = \mu_\Gamma$ where μ_Γ is the aggregation rule generated by Γ. Let a, $b \varepsilon A$, and suppose $a \varepsilon f(p)$ for some profile p. To complete the proof it is sufficient to show that $a \varepsilon f(q)$ for any profile q such that the transformation from p to q does not demote a in any individual's weak ordering.

From $a \varepsilon f(p)$ it follows that $I/\{i|aR_ib\} \not\varepsilon \Gamma_I$ for all $b \varepsilon A$ with $b \neq a$. Otherwise, there would be a set $C \subseteq I$ such that for some $e \varepsilon A$ $C = \{i|eP_ia\} \varepsilon \Gamma_I$ and then ePa so that $a \not\varepsilon f(p)$. Suppose q is a profile with the property that the transformation $p \to q$ does not demote q in any individual's weak ordering. It follows that when p changes to q, $\{i|bP_ia\}$ will not expand for any $b \varepsilon A$. Because Γ_I is a monotonic game, $D \not\varepsilon \Gamma_I \Rightarrow E \not\varepsilon \Gamma_I$ whenever $E \subseteq D$. Therefore, there will be no $b \varepsilon A$ such that $\{i|bP_ia\} \varepsilon \Gamma_I$ under profile q since no such set is in Γ_I under profile p. Since f is a simple social function, it follows from $\{i | bP_i a\} \notin \Gamma_I$ under profile q for all b ε A that there will be no c ε A such that $\{i | cP_i a\} \in \Gamma_J$ for any $J \subseteq I$ under profile q. Thus, under profile q there will be no c ε A such that cPa, and a ε f(q).

Q.E.D.

The following theorem establishes a condition on direct sums of simple games equivalent to Maskin's NVP condition.

Theorem 6: A neutral monotonic social function generated by Γ, a direct sum of simple games, will have the property NVP if and only if there is no $J \subseteq I$ such that Γ_J is an ultrafilter.

Proof: (a) Necessity: If there is a $J \subseteq I$ such that Γ_J is an ultrafilter, then there is an i ε J such that $\{i\} \in \bigcap \Gamma_J$. Since Γ_J is a filter, $C = \bigcap \Gamma_J \in \Gamma_J$. It is easy to show that C must be a singleton. If it is not, then no singleton is in Γ_J and all coalitions of size $|J| - 1$ are in Γ_J by the ultrafilter property that $D \notin \Gamma_J \Rightarrow J/D \in \Gamma_J$. Then $\bigcap \Gamma_J = \emptyset$ since the intersection of all coalitions of size $|J| - 1$ is empty. This contradicts Γ_J being a filter.

Assume there is a $J \subseteq I$ such that Γ_J is an ultrafilter and let {i} be the singleton member of Γ_J. Consider a profile p such that for a, b ε A $aP_i b$, $J/\{i\} = \{j | bP_j a\}$ and $I/J = \{k | bI_k a\}$. Now it will be the case that b ∉ f(p) but, for all j ≠ i, $bR_j a$. This contradicts NVP.

(b) Sufficiency: Suppose f does not satisfy NVP. Then there exist a, b ε A such that bPa and $aR_j b$ for all j ε I except possibly for some i ε I. If $aR_i b$, then bPa when there is no k ε I such that $bP_k a$. But since Γ is a direct sum of simple games, $\emptyset \notin \Gamma_J$ for any $J \subseteq I$. Thus, bPa is impossible unless $bP_k a$ for at least one k ε I.

Suppose then that $bP_i a$. Let $C = \{j | aP_j b\}$ and $D = \{i\} \cup C$. Since bPa, $\{i\} \in \Gamma_D$. Since $\{i\} \in \Gamma_D$, if $i \in E \subseteq D$ then $E \in \Gamma_D$ since Γ_D is a monotonic game. Furthermore, any $F \subseteq D$ such that $i \notin F$ will be such that $D/F \in \Gamma_D$ and thus $F \in \Gamma_D^*$. It follows that (1) $\bigcap \Gamma_D = \{i\} \in \Gamma_D$ and (2) for any $G \subseteq D$ either $G \in \Gamma_D$ or $D/G \in \Gamma_D$ since one of G and D/G must contain i. By (1) and (2), Γ_D is an ultrafilter.

Q.E.D.

Given Theorems 5 and 6, fully Nash implementable neutral social functions and fully strong Nash implementable neutral social functions can be partially characterized.

Theorem 7. A neutral social function f will be either fully Nash implementable or fully strong Nash implementable only if f is a simple neutral monotonic social function.

Proof: Theorem 2 in Maskin (1977) and Theorem 1 in Maskin (1979) show that Maskin monotonicity is a necessary condition both for full Nash implementation and for full strong Nash implementation of social choice correspondences over finite alternative sets. Nothing in the proof of either result depends on the finiteness of the alternative set, so these results also hold for CCE subset alternative sets. By Theorem 5, a neutral social function that is Maskin monotonic must be a simple neutral monotonic social function.

Q.E.D.

Theorem 8: If a neutral binary decision rule f is a simple neutral monotonic social function generated by a direct sum of simple games Γ and if Γ_I is not an ultrafilter, then f is fully Nash implementable.

Proof: A simple neutral monotonic social function f is Maskin monotonic by Theorem 5. Let Γ be the direct sum of simple games that generates f. If Γ_I is not an ultrafilter, then no Γ_J for $J \subseteq I$ is an ultrafilter since f is simple. By Theorem 6, f satisfies NVP. Theorem 5 in Maskin (1977) shows that Maskin monotonicity and NVP are jointly sufficient for a social choice correspondence to be fully Nash implementable over finite alternative sets. Theorem 4 in this article shows that Maskin monotonicity and NVP are jointly sufficient for a social choice correspondence to be fully Nash implementable over CCE subset alternative sets.

Q.E.D.

Theorem 9. A neutral monotonic social function over a finite alternative set generated by a direct sum of simple games Γ is fully strong Nash implementable only if for some $J \subseteq I$, Γ_J is an ultrafilter.

Proof: By Theorem 2 in Maskin (1979) no social choice correspondence

satisfying NVP is fully strong Nash implementable. By Theorem 6 in this article, a neutral monotonic social function f will have the property NVP if and only if Γ, the direct sum of simple games that generates f, contains no simple game that is an ultrafilter.

Q.E.D.

For the case of neutral social functions over finite alternative sets, a complete characterization of fully strong Nash implementable rules is possible using Theorems 7 and 9.

Theorem 10. A neutral social function f over a finite alternative set will be fully strong Nash implementable if and only if f is a simple neutral monotonic social function where the set of all winning coalitions under f is an ultrafilter.

Proof: (a) Necessity: Suppose f is a neutral social function that is fully strong Nash implementable. By Theorem 7 f must be a simple neutral monotonic social function. If the direct sum of simple games that generates f is Γ, then Γ_I must be the set of all winning coalitions under f. By Theorem 9, there is a concerned set $J \subseteq I$ such that Γ_J is an ultrafilter. It follows that $\{i\} \in \Gamma_J$ for some $i \in I$. Since f is simple, $\{i\} \in \Gamma_I$, and Γ_I is therefore an ultrafilter.

(b) Sufficiency: If f is a simple neutral monotonic social function where the set of winning coalitions under f is an ultrafilter, it is straightforward to construct a decision mechanism that fully implements f by strong Nash equilibria. Since f is a simple neutral monotonic social function, it is generated by a direct sum of simple games Γ, and Γ_I contains all coalitions that win in any concerned set. Since Γ_I is an ultrafilter, by the argument in the proof of Theorem 6 there exists an individual i such that $\{i\} \varepsilon \Gamma_I$ and $\cap \Gamma_I = \{i\}$. It follows that for any two x, y ε A xPy if and only if xP_iy and xIy if and only if xI_iy. (When xI_iy, then neither $\{j \varepsilon I | xP_jy\}$ nor $\{k \varepsilon I | yP_kx\}$ is a winning coalition since Γ_I contains all winning coalitions and $\cap \Gamma_I = \{i\}$.) The choice set therefore consists of all the alternatives at the top of the dictator i's weak ordering.

Let the decision mechanism consist of each individual specifying a single "first choice" with the social choice being the dictator's specified first choice. It is easy to show that the set of strong Nash equilibria for this decision mechanism is precisely the choice set. Suppose the dictator specifies an alternative x from the top of his or her weak ordering as his or her first choice and therefore as the outcome of the decision mechanism. Then the dictator cannot be part of any group all of whose members would be better off by jointly altering their strategies since the dictator does not prefer any alternative to x. But no group without a dictator can change the social outcome by changing their strategies. In addition, if the dictator did not specify an alternative at the top of his or her weak ordering as a strategy, then the dictator can do better by switching his or her strategy to such an alternative. Thus no strategy vector where the dictator does not specify a first choice at

the top of his or her weak ordering can be a strong Nash equilibrium.

Q.E.D.

Theorem 7 combined with Theorem 2 in Strnad (1982) provide a link between those neutral social functions that are continuous-valued and those that are fully Nash implementable or fully strong Nash implementable. Consider the following corollary:

<u>Corollary 1</u>. A neutral social function f over a CCE subset alternative set will be fully Nash implementable or fully strong Nash implementable only if f is continuous-valued for all profiles of continuous-valued individual weak orderings.

<u>Proof</u>: By Theorem 7, a neutral social function f is fully Nash implementable or fully strong Nash implementable only if f is a simple neutral monotonic social function. By Theorem 2 in Strnad (1982) a neutral monotonic social function f will be continuous-valued for all profiles of continuous-valued individual weak orderings if and only if f is simple.

Q.E.D.

Ferejohn, Grether and McKelvey [7] prove some characterization results for Nash implementable and strong Nash implementable social choice correspondences. Analogous results for fully Nash implementable and fully strong Nash implementable neutral social functions are straightforward corollaries of Theorem 7. One Ferejohn, Grether and McKelvey result is their Theorem 1. In the terminology of this article that result is that when A is a finite set, $|I| \leq |A|$ and

$|A| \geq 3$, then a social choice correspondence f with a nonempty choice set for all profiles of weak orderings is neither Nash implementable nor strong Nash implementable if f is minimally democratic.[2] The following corollary provides an analog to that result for neutral social functions that are fully Nash implementable or fully strong Nash implementable.[3]

<u>Corollary 2</u>: When A is finite and $|A| \geq |I|$, a neutral social function that is fully Nash implementable or that is fully strong Nash implementable cannot be minimally democratic and have a nonempty choice set for all profiles of weak orderings.

<u>Proof</u>: Let f be a neutral social function. By Theorem 7, if f is fully Nash implementable or fully strong Nash implementable, then it is a simple neutral monotonic social function. Suppose that Γ is the direct sum of simple games that generates f. Since f is simple, Γ_I contains all the winning coalitions under f.

The following argument demonstrates that if f is minimally democratic, Γ_I cannot be a prefilter. Suppose Γ_I is a prefilter and $i \in I$ is in the collegium. Choose a profile p such that for all $j \in I/\{i\}$ there is an $x \in A$ such that xP_jz for all $z \in A/\{x\}$. Under p let i have preferences such that for some $y \in A$ yP_iz for all $z \in A/\{y\}$ and xP_iz for all $z \in A/\{x, y\}$. Since f is minimally democratic, f(p) = $\{x\}$ is required. But since $i \in \cap\Gamma_I$ and Γ_I contains all coalitions that win in any concerned set, $y \in f(p)$.

If Γ_I is not a prefilter, then Γ is not a direct sum of prefilters. By Theorem 4 in Blau and Brown (1978) f is not a social decision function when $|A| \geq |I|$ unless f is generated by a direct sum of prefilters. As a result, profiles can be constructed such that there is a social cycle over all the alternatives in a finite set. To see this, suppose the alternative set is $\{x_1, x_2, \ldots, x_m\}$ and consider the chain of coalitions $C_1, \ldots, C_m$ such that $\bigcap_{i=1}^{m} C_i = \emptyset$, $C_i \in \Gamma_I$ and all members of coalition C_i prefer x_i to x_{i+1} (where $x_{m+1} = x_1$). Since each C_i is a winning coalition, there is a social cycle, $x_1Px_2Px_3 \ldots Px_nPx_1$. Blau and Brown (1978) show that there is a profile of weak orderings consistent with this situation. That the condition $\bigcap_{i=1}^{m} C_i = \emptyset$ for $C_i \in \Gamma_I$ is possible follows from the following argument. Since Γ_I is not a prefilter, there is some set S of coalitions in it that has empty intersection. Make this a minimally-sized set S' by excluding the largest number of coalitions from S such that the intersection is still empty. Each coalition in S' must uniquely exclude at least one member of I from the intersection. Otherwise that coalition would be extraneous. It follows that S' has at most n coalitions in it where $|I| = n$. But since $n \leq m$, it is possible to choose m coalitions C_i such that $\bigcap_{i=1}^{m} C_i = \emptyset$.

Now if Γ_I is not a prefilter, for some $p \in W^I$ it will be the case that $f(p) = \emptyset$. This contradicts the requirement that the choice set be nonempty for all profiles in W^I.

Q.E.D.[4]

The arguments in the proof of Corollary 2 indicate the value of the characterization result in Theorem 7. Starting with neutral social functions that result yields simple neutral monotonic social functions. Given simple neutral monotonic social functions, all the machinery of Blau and Brown (1978) and Strnad (1982) can be used to assess whether and how implementability is affected by various restrictions. In particular, the proof of Corollary 2 rests solely on the fact that when A is finite and $|A| \geq |I|$ no neutral monotonic social function can simultaneously be minimally democratic and have a nonempty choice set for all profiles of weak orderings. No aspect of implementation such as the nature of the equilibria sets is used in the proof.

FOOTNOTES

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This article is derived from a chapter of my Ph.D. dissertation, Strnad (1982). I have profited greatly from the suggestions and guidance of Donald Brown in this work. Any remaining errors are solely my responsibility.

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1. This definition is taken from Kramer (1977). Type I preferences have (hyper)spherical indifference curves with a bliss point in the center of the (hyper)sphere.

2. Ferejohn, Grether and McKelvey's Theorem 1 also encompasses implementation by equilibrium concepts "in between" Nash equilibrium and strong Nash equilibrium. In particular, they define a k-equilibrium as one where no coalition of size less than or equal to k can make all of its members better off by jointly changing their strategies. Thus, Nash equilibrium is 1-equilibrium and strong Nash equilibrium is n-equilibrium where $|I| = n$. In its full generality Ferejohn, Grether and McKelvey's Theorem 1 asserts that no minimally democratic social choice correspondence is implementable by a set of k-equilibria for

$k \in \{1,2,\ldots,n\}$.

3. No more than an analog is presented here for two reasons. First, Ferejohn, Grether and McKelvey's Theorem 1 is not limited to neutral social functions but applies to the more general class of all social choice correspondences. Second, the characterization results in this chapter rest heavily on the exploitation of the property of Maskin monotonicity. Ferejohn, Grether and McKelvey do not use Maskin monotonicity. They use a form of monotonicity that focuses on equilibrium set elements instead of choice set elements. Thus, by the Lemma in their article for any $k \in \{1,2,\ldots,n\}$ if x is a k-equilibrium under one profile and if the profile is changed in a way that does not demote x in anyone's weak ordering, then x will be a k-equilibrium under the new profile. (See note 2, supra, for a definition of k-equilibrium.)

   A social choice correspondence could have this monotonicity property (call it "FGM monotonicity") and yet not be Maskin monotonic when implementation rather than full implementation is studied. Under implementation the equilibrium set may not contain all choice set alternatives, and consequently even given FGM monotonicity there may be a choice set alternative y that falls out of the choice set by a change of profiles that does not demote y in any individual's weak ordering.

4. For the case of full strong Nash implementation this proof can be greatly simplified by using a result already in the literature.

Specifically, by Corollary 4.4 in Ferejohn and Grether (1981) when $|A| \geq |I|$ and a social choice correspondence f is fully strong Nash implementable, then the collection of "prevalent coalitions" for f is a prefilter. A coalition is prevalent when the choice set consists of an alternative x alone for all profiles where all members of the coalition prefer x to all other alternatives.

Ferejohn and Grether's result applies in this situation because a neutral social function is a social choice correspondence. Given a neutral social function f that is fully strong Nash implementable, Theorem 7 requires f to be a simple neutral monotonic social function. Given that f is simple, the set of prevalent coalitions for f is Γ_I where Γ is the direct sum of simple games that generates f. Ferejohn and Grether's result indicates that Γ_I must be a prefilter.

Despite the existence of that result, this proof goes on to consider the case where Γ_I is not a prefilter. The purpose of such an approach is to make clear where the restriction $|A| \geq |I|$ is used.

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